Math 210

Quiz #2, April 26, 2004

" Oh! the little more, and how much it is!
And the little less, and what worlds away!" [Robert Browning]

1. (20 pts.) Let f be the function defined by

$$f(x) = \left\{ \begin{array}{ll} x^4 \sin(\frac{1}{x^2}), x \neq 0 \\ 0, \quad x = 0. \end{array} \right\}$$

Prove that f is differentiable, and that f' is continuous but not bounded on $(-\infty, \infty)$.

2. (20 pts.) Let c_n be the sequence defined by

$$c_n = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} - 2\sqrt{n}, n = 1,2,3,\ldots .$$

Prove that c_n is monotone and bounded.

3. (20 pts.) Let $f : R \to R$ be a differentiable function and suppose that $|f'(x)| \leq 0.49$ for all $x \in R$. Prove that the equation $f(x) = \frac{2x+\sin x}{2}$ has a unique solution in R.

4. (20 pts.) Let $f : [a,b] \to R$ be a given function and suppose that, for each $x \in [a,b]$, $\lim_{t\to x} f(t)$ exists. Prove that f is bounded on $[a,b]$. **Hint**: Denote the limit of f at x by L_x. i.e. $\lim_{t\to x} f(t) = L_x$, and show that, for each $x \in [a,b]$, there is a nbhd V_x where f is bounded. Be sure to specify a bound in V_x.

5. (20 pts.) Let $f : [a,b\} \to R$ be continuous on $[a,b]$ and differentiable on (a,b). Suppose that $f(a) = 0$, and there is a constant $M > 0$ such that $|f'(x)| \leq M|f(x)|$ for all $x \in (a,b)$.

(i) If $c \in (a,b)$ and $a < x < c$, prove that $|f(x)| \leq MM_c|x-a| \leq MM_c|c-a|$, where $M_c = \sup_{a \leq x \leq c} |f(x)|$.

(ii) If $c \in (a,b)$ and $M_c \neq 0$, prove that $1 \leq M|c-a|$.

(iii) Prove that $f(x) = 0$ for every $x \in [a,b]$.

$$C_n = \frac{1}{n^3} + \frac{1}{(n+1)^3} + \cdots + \frac{1}{(2n)^3}$$

$$c_{n+1} - c_n = \frac{1}{(2(n+1))^3} + \frac{1}{(2n+1)^3} - \frac{1}{n^3} < \frac{1}{8n^3} + \frac{1}{8n^3} - \frac{1}{n^3} < \frac{1}{4n^3} - \frac{1}{n^3} < 0$$

in fact $\quad \frac{n+1}{8n^3} < c_n < \frac{n+1}{n^3} \quad c_n \to 0$